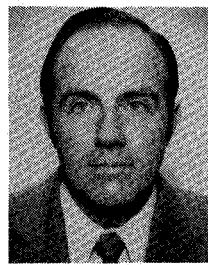


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Numerical Analysis of Open-Ended Coaxial Lines

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Abstract—Numerical methods are applied in the analysis of coaxial structures used as sensors for *in vivo* permittivity studies of biological substances. The methods used for the solution of the resulting static conductor-dielectric problems are the Finite Element Method (FEM) and the Method of Moments (MOM) applied to a pair of coupled integral

equations. A linear model which relates the sample permittivity to the fringing field capacitance of the sensor is discussed and values of the model parameters are calculated for different types of sensors.

I. INTRODUCTION

OPEN-ENDED coaxial lines have been used extensively as sensors for permittivity measurements of biological substances in recent years [1]. Their simple geometry and small size (potentially as small as 0.5-mm diameter) makes them suitable for *in vivo* measurements as

Manuscript received June 25, 1982; revised December 29, 1982. This work was supported by grants from the National Sciences and Engineering Research Council of Canada and Health and Welfare Canada.

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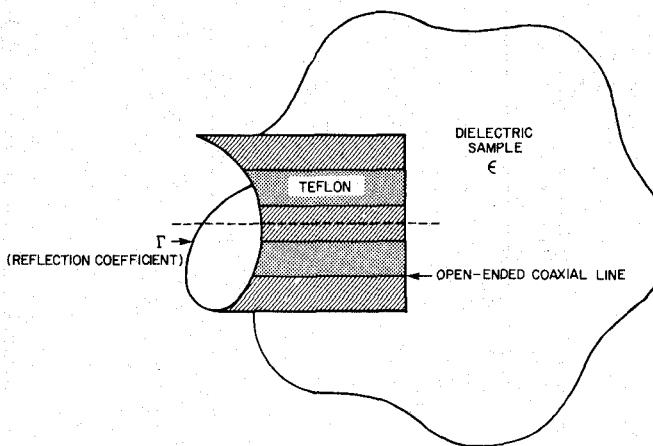


Fig. 1. Open-ended coaxial sensor for dielectric measurement.

well as measurement of the spatial distribution of permittivity. Other advantages of the open-ended coaxial line over other sensor configurations are: a broad frequency range; compatibility with time-domain, applicability to frequency-domain, and resonant measurement techniques; and ease of fabrication.

When used as a sensor for dielectric measurement, the open end of the line is inserted into the sample (as in Fig. 1), and the input reflection coefficient (or input admittance) is measured at a specific frequency and temperature. Various methods have been used to relate the reflection coefficient data to the dielectric properties of the sample. These range from analytical/graphical methods [2], to equivalent circuit approaches and interpolation methods [3].

A solution of the scattering from the open end of a coaxial line in contact with a lossy dielectric was presented by Mosig *et al.* [2]. Nomograms of reflection coefficient versus relative permittivity of the half-space medium were constructed, from which ϵ' and ϵ'' could be calculated for a given reflection coefficient. A major limitation of this approach is that extensive nomograms are required at each measurement frequency. Also, the numerical computations required to generate the nomograms become increasingly time consuming for high permittivities such as those encountered in biological substances at low frequencies [2].

Several authors have made use of a lumped equivalent circuit, relating the admittance of the sensor to the permittivity. This approach has the advantage that closed-form expressions for ϵ' and ϵ'' as a function of the reflection coefficient can be derived and used in automatic network analyzer routines. Burdette *et al.* [3] used an equivalent circuit consisting of a single lumped-shunt-capacitance whose effective value was equal to the fringing capacitance of the open-ended line in air multiplied by the sample permittivity. An additional shunt capacitance whose value is independent of the sample permittivity was added in [4] to account for fringing inside the coaxial line. Also, the effect of radiation from the open end on the equivalent circuit was treated by Stuchly *et al.* [5] and used in measurements by Brady *et al.* [6]. In all of the references

pertaining to the lumped circuit approach, the equivalent circuit parameters of the sensors were either measured directly or inferred from measurements on known dielectrics. Also, the assumption was made that the fringing capacitances are linearly proportional to the permittivity and independent of frequency.

The purpose of this work is to use numerical methods to investigate the behavior of the fringing capacitance of coaxial sensors, shown in Fig. 1, as a function of the sample permittivity. In particular, the two-capacitance or linear model, relating the net fringing capacitance to the permittivity, is examined.

Since it is known [7]–[9] that the fringing capacitance for the homogeneous case is constant from dc to microwave frequencies (for all practical sizes of line), only the static fringing capacitances are calculated in the present work. The two numerical methods selected to solve the resulting static conductor–dielectric problems are the Finite Element Method (FEM) and the Method of Moments (MOM) applied to a pair of coupled integral equations. Since all solutions are for the static case, only lossless dielectric are modelled.

II. THEORY

A. Capacitance Model

For the purposes of presentation of the numerical results, a linear model relating the net fringing capacitance to the permittivity ϵ is assumed. The fringing capacitance may be written as

$$C(\epsilon) = C_f + \epsilon C_0 \quad (1)$$

where ϵ is the relative permittivity of the sample occupying the space outside the line (Fig. 1).

This is the form of $C(\epsilon)$ used by a large number of investigators [1]. The constant term C_f may be considered to represent storage of energy in the fringing fields inside the line while the linear term ϵC_0 represents energy storage in the dielectric.

For the case where the sample dielectric is air ($\epsilon = 1$), the net fringing capacitance is equal to the algebraic sum of C_f and C_0

$$C(1) = C_f + C_0 = C_T \quad (2)$$

where the value of $C(1)$ is designated the total capacitance C_T . This quantity is readily measured, or, for the homogeneous case (i.e., for an air line), it may be calculated from a formula given in [8].

Numerical methods are used to calculate values of the net fringing capacitance $C(\epsilon)$ for each assumed value of ϵ . From this data, values of C_f and C_0 can be calculated for each ϵ by solving the simultaneous linear equations (1) and (2). In general, both C_f and C_0 will be functions of ϵ if $C(\epsilon)$ varies nonlinearly with ϵ ; however, if a range of ϵ exists where C_f and C_0 , calculated in the above manner, are constant, the linear model will be approximately valid over this range.

B. Numerical Methods

1) *Finite Element Method*: The two-dimensional FEM was used to solve Laplace's equation indirectly in the rotationally symmetric region of Fig. 2, representing the open end of a coaxial line. The region which is partially bounded by Dirichlet and homogeneous Neumann boundaries was divided into triangular elements. In each element, the permittivity ϵ_t is known and the unknown potential ϕ is approximated by a polynomial trial function with constant coefficients. The trial functions were substituted into the variational expression

$$F = \int_R \epsilon (\nabla \phi)^2 dR \quad (3)$$

$$= \sum_t^{\text{No. T.}} \epsilon_t \int_{R_t} (\nabla \phi)_t^2 dR_t \quad (4)$$

which is proportional to the stored energy in the system, and (4) was minimized with respect to the unknown constant coefficients. This procedure produced a system of algebraic equations for the unknown coefficients which can be solved by standard methods.

It can be shown that the continuity of the normal flux between adjacent dielectric regions (interface conditions) are satisfied as a result of minimizing the variational expression [4], [10].

Since the problem is unbounded in the positive z -direction, an approximate Neumann boundary representing fringing electric field line is used to close the region of solution. The position of the approximate boundary is first assumed and later, as solutions are run, it is further refined. The criterion used for determining a sufficient size for the half-space region in Fig. 2 was the convergence of the total stored energy.

2) *Method of Moments*: The MOM was used to solve for the unknown charge distribution on the conductor and dielectric interface surfaces shown in Fig. 2. The coupled integral equations which relate the free and bound surface charge densities $\sigma(s)$ residing on conductors and dielectric interfaces, respectively, to the potential distribution $\phi(s)$ are given by [11]

$$\int_{S_c + S_I} \sigma(s') G(s|s') ds' = \phi(s), \quad s \text{ on } S_c \quad (5)$$

$$\frac{\epsilon_1 + \epsilon_2}{2} \sigma(s) + (\epsilon_1 - \epsilon_2) \int_{S_c} \sigma(s') \frac{\partial G}{\partial n}(s|s') ds' = 0, \quad s \text{ on } S_I \quad (6)$$

where $G(s|s')$ is the free-space potential Green's function and S_c and S_I denote conductor and interface surfaces, respectively.

The two permittivities ϵ_1 and ϵ_2 correspond to the two dielectric regions and the normal n is directed from region 1 to region 2 as in Fig. 3.

In rotationally symmetric systems, the free-space Green's function and its normal derivative may be written in terms of the complete elliptic integral of the first kind and its derivative [12].

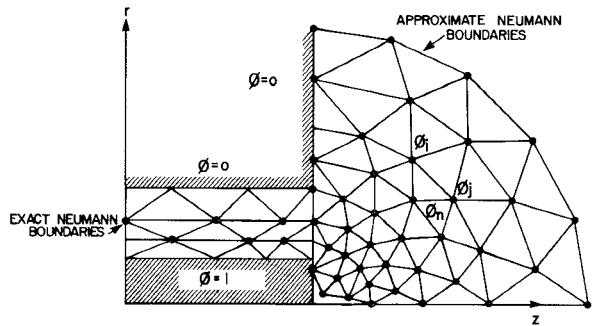


Fig. 2. Open-ended coaxial line with groundplane showing triangulation of region for solution by FEM.

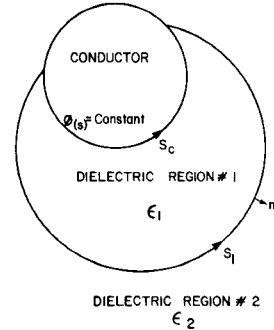


Fig. 3. Generalized configuration of an unbounded conductor-dielectric problem

The solution of (5) and (6) for the unknown surface charge distribution $\sigma(s)$ with a known potential distribution $\phi(s)$ proceeds by dividing the surface into subsections and assuming a uniform charge density of unknown amplitude on each subsection. Next, the discretized integral equations are enforced at the midpoint of each subsection, producing a system of algebraic equations for the unknown charge-pulse amplitudes. This is equivalent to using pulse expansion and Dirac weighting functions in the MOM [13].

In order to approximate an infinite line in the negative z -direction (see Fig. 2), the method of images was used to solve for the charge on the real conductors and interfaces and their mirror image about $z = 0$. This insures an almost uniform charge distribution inside the line far from the aperture. Numerically, this involved adding a term to the Green's function to account for the presence of image charges; however, it does not affect the number of unknowns to be solved.

For both the MOM and the FEM, the resulting capacitance possesses a component due to the TEM capacitance of an infinitely long line. The TEM capacitance, given by

$$C_{\text{TEM}} = \frac{2\pi\epsilon_{\text{line}}\Delta l}{\ln(b/a)} \quad (7)$$

where Δl is the length of line, must be subtracted from the result in order to yield the net fringing capacitance.

III. NUMERICAL RESULTS

For comparison with the analytical expression for the total fringing capacitance found in [8], the configuration of a 50- Ω air line opening into a groundplane and air-dielec-

TABLE I

COMPARISON OF TOTAL CAPACITANCE OBTAINED FROM FEM AND MOM

	Ref. [9]	FEM	MOM	Measured [4]
C_T	4.295	4.311	4.076	3.92
$\epsilon_0(b-a)$				

tric was solved using both methods. The ratio of outer- to inner-conductor radii for this type of line is $b/a = 2.3$ and the groundplane radius was taken to be 3 times the outer-conductor radius. Using the FEM, the region of solution shown in Fig. 2 was divided into 125 triangular elements with 80 nodes. The necessary area of the half-space required to contain most of the stored energy was found to have a radius of approximately 2.5 times the outer-conductor radius. This area was determined by progressively increasing the radius of the quadrant until the change in stored energy became less than 1 percent.

In the MOM solution, the conductors were divided into 65 subsections. 16-point Legendre-Gauss quadrature was used to integrate the Green's function for calculation of the matrix elements except the diagonal ones which were calculated using a combination of analytic and numerical integration [15].

Both programs were written in double-precision Fortran, and Gauss-elimination [17] was used for the solution of the resulting system of equations. Table I compares the values of fringing capacitance from both methods with the value computed from the low-frequency asymptotic formula in Marcuvitz [8].

The values of capacitance given in Table I and elsewhere are normalized to the free-space permittivity ϵ_0 and aperture dimension $(b-a)$. Thus the results in Table I are unitless quantities and pertain to any size of open-ended 50- Ω air line.

Calculations of the fringing capacitance of 50- Ω teflon dielectric lines (identical to Fig. 1) were performed for a range of half-space permittivities $1 < \epsilon < 60$. These lines are available commercially in sizes ranging from 0.5 mm (0.02") to 6 mm (0.25") in diameter. The ratio of outer-conductor inner-radius to inner-conductor radius is $b/a = 3.27$ and outer-conductor outer-radius to inner-conductor radius is $c/a = 3.95$, while the line dielectric constant is $\epsilon_{\text{line}} = 2.05$.

For the FEM, 156 triangles and 96 nodes were used, while in the MOM the conductors and interface were divided into 66 and 12 subsections, respectively. Values for the normalized fringing capacitance $C(\epsilon)/\epsilon_0(b-a)$ as a function of ϵ are presented in Table II, while values of the two linear model parameters $C_f/\epsilon_0(b-a)$ and $C_0/\epsilon_0(b-a)$ are shown plotted versus ϵ in Fig. 4. For comparison, measured values are also indicated where applicable.

IV. DISCUSSION

From the results in Table I corresponding to an open-ended air line with a groundplane, the two numerical methods gave values of normalized fringing capacitance within 5 percent of the value computed from the formula in [8] with the FEM giving the closest value. On the other

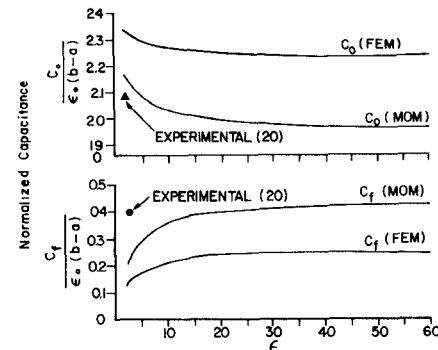
Fig. 4. Normalized C_f and C_0 versus ϵ for 50- Ω teflon dielectric lines.

TABLE II
NORMALIZED FRINGING CAPACITANCE VERSUS ϵ (FEM AND MOM)

ϵ	$C(\epsilon)/\epsilon_0(b-a)$		
	FEM	MOM	Measured
1.0	2.48	2.38	2.42
2.0	4.83	4.56	----
5.0	11.67	10.72	----
10.0	22.88	20.62	----
20.0	45.15	40.11	----
40.0	89.60	78.85	----
60.0	134.02	117.53	----

hand, the MOM result is closer to the measured value reported for a 14-mm air line [4].

For the case of an open-ended teflon-filled line, the MOM gave consistently smaller values of fringing capacitance over the whole range of ϵ (Table II). When the two parameters of the linear model C_f and C_0 are calculated according to Section II-A using the data in Table II, it is seen that both parameters vary with ϵ , especially for small ϵ . Also, as a result of the way in which they were defined and the consistently larger values of $C(\epsilon)$ produced by the FEM, it may be seen (Fig. 4.) that the FEM gives larger values of C_0 and smaller values of C_f than the MOM over the range of ϵ . In Fig. 4, the values of C_f and C_0 computed from the MOM data for $\epsilon = 2$ are closer to the measured values reported in [20] than for the FEM.

In view of the close correspondence between the MOM results and measured values, it is felt that the results obtained from the MOM are more accurate than the FEM. Although no quantitative analyses of the errors in both methods were performed, several observations during the course of the work corroborate this conclusion. It was found that the FEM converged rather slowly as the number of elements increased, whereas the MOM converged quickly with an increasing number of subsections. Also, the error introduced by truncating the region of solution in the FEM may lead to a cancellation of errors since, for the functional used in this work, the FEM should give an upper bound for the capacitance.

In terms of the linear model for the fringing capacitance as a function of the permittivity, it may be seen from Fig. 4 that the use of such a model is a good approximation for large permittivities since C_f and C_0 are relatively constant in this range. For small permittivities, say $\epsilon < 10$, the linear

model is not such a good approximation since C_f and C_0 vary with ϵ , although the percentage variation in C_0 is relatively small. This fact together with the fact that C_f is small compared to C_0 enables the linear model to be used for all practical purposes for small values of permittivity.

V. CONCLUSION

The numerical analysis of the static fringing capacitance of an open-ended coaxial line in contact with a dielectric has been presented. This structure finds practical applications as a sensor for *in vivo* permittivity measurements at radio and microwave frequencies. The numerical values of the fringing capacitance, obtained from an application of the FEM and the MOM, were used to examine the validity of the two-capacitance or linear model relating the sample permittivity to the fringing capacitance. The results showed the linear model to be a good approximation for large permittivities such as those encountered in biological materials. The two methods also agreed with measured values of fringing capacitance reported in the literature.

For smaller permittivities (i.e., $\epsilon < 10$), a more accurate model is required as evidenced by the variation of the linear model parameters with ϵ over this range.

ACKNOWLEDGMENT

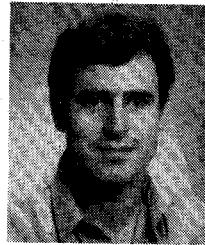
Valuable and stimulating discussion with Dr. G. I. Costache from Bell-Northern Research, Ottawa, are gratefully acknowledged.

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